Monochromatic Triangles in $\mathbb{E}^2$.

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1 Nondegenerate triangles in the plane

In 1973, Erdős, Graham, Montgomery, Rothschild, Spencer, and Straus published the three seminal papers of Euclidean Ramsey theory ([1], [2], [3]). In the third of these, they consider this question: if the points in $\mathbb{E}^2$ are partitioned into two sets – say, red and blue – then which sets must occur monochromatically (that is, in one of the parts), and which can be avoided?

More formally, for a finite set $X \subset \mathbb{E}^2$, let *Cong(X)* be the set of all subsets of $\mathbb{E}^2$ which are congruent to X under some Euclidean motion (including reflection). Fixing a finite set $X \subset \mathbb{E}^2$, consider the set of all maps $\chi : \mathbb{E}^2 \to \{red, blue\}$. If in every case there is some X' in $\chi^{-1}(red)$ or in $\chi^{-1}(blue)$ with $X' \in Cong(X)$, we say that X cannot be avoided by two colors – there is always a monochromatic copy of X, regardless of the coloring χ. This notion extends in the obvious way to more than two colors.

It is easy to see that if X consists of two points, then we cannot avoid it with two colors: let d be the distance between the two points, and try to 2-color the vertices of any equilateral triangle of side d. In [3] the authors show that if X is an equilateral triangle of side d (by a triangle, we mean the set of its vertices), then it can be avoided, by coloring the plane with alternating horizontal red and blue strips of width $\sqrt{3}d/2$, each half-open at the top. There are various triangles that are known to be impossible to avoid with two colors; a list of some families of these is given in [3], and L. Shader has shown in [4] that all right triangles also belong on this list.

Conjecture 1. *[3] For any non-equilateral triangle T, every 2-coloring of $\mathbb{E}^2$ contains a monochromatic copy of T.*

This is still open, and we make no direct progress toward Conjecture 1 here. Instead, we note that in [4], as a lemma to the main result, we have:

Lemma 1. *For any real number a and 2-coloring of the plane, there is a monochromatic equilateral triangle of side ka, for some* $k \in \{1, 3, 5, 7\}$.

As a special case of Theorem 9 in [1], we have

Theorem 1. *If* T *is a set of three noncollinear points and* χ *is any 2-coloring of* $\mathbb{E}^2$*, then* χ *contains a monochromatic congruent copy of* T*,* $2T$*, or* $\sqrt{3}T$ *(where* kT *is just the triangle* T *scaled by a factor of* k*).*

In this note we present a similar result.

1.1 The main result

Theorem 2. *If* T *is a set of three noncollinear points and* χ *is any 2-coloring of* $\mathbb{E}^2$*, then* χ *contains a monochromatic* **translate** *of* T*,* $2T$*,* $3T$*, or* $4T$*.*

Proof. Consider the triangle $4T$ built from copies of T, as in Figure 1. Note that this orientation was chosen to facilitate the proof, and below we will refer to the "top" vertex, etc., casually; of course T need not actually be oriented this way.

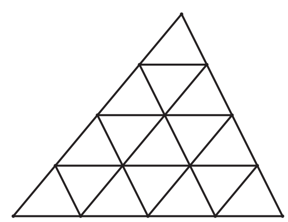

Figure 1: The triangle $4T$ formed from T.

Suppose by way of contradiction that we can color the 15 vertices of this diagram without producing a monochromatic T, $2T$, $3T$, or $4T$. Then the outermost vertices cannot be the same color (our two colors here will be black and white). Without loss of generality, color the top and leftmost vertices black, and the rightmost white. This leads us to Figure 2, which includes vertex labels that we will use below.

Note that in Figure 2, vertex B must be white, otherwise vertices D and E would both be forced to be white, producing a white $2T$. Vertices A, B, and C cannot all be white, because then F and G would be forced to be black, and it would be impossible to color H. Note that this logic applies to any three consecutive vertices.

Thus, one of A and C is black; by symmetry, we may arbitrarily choose A. This forces I to be white, leaving us at Figure 3.

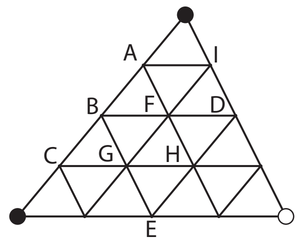

Figure 2: Coloring the outermost vertices.

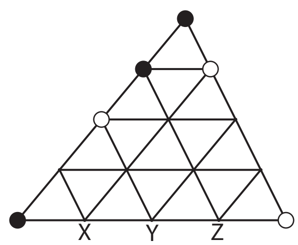

Figure 3: After coloring some more vertices.

Now to avoid a monochromatic $3T$, vertex X must be black and vertex Z must be white. It is now impossible to color vertex Y without producing three consecutive like-colored vertices, so the proof is complete. □

This leads to another result if we consider congruence instead of simply translation.

Corollary 1. *If T is a set of three noncollinear points and χ is any 2-coloring of $\mathbb{E}^2$, then χ contains a monochromatic congruent copy of T, $2T$, or $3T$.*

Proof. Fix χ and suppose there is no monochromatic congruent copy of T, $2T$, or $3T$. Then if the triangular lattice from Theorem 2 is placed onto the plane in any position, in any orientation, the outermost vertices will be monochromatic (otherwise that lattice would be colored to avoid any monochromatic T, $2T$, $3T$, or $4T$, a contradiction to Theorem 2). This easily implies that the whole plane is monochromatic, a contradiction. □

1.2 Conclusion

While the results above do not lead to any new forced monochromatic triangles among 2-colorings, observe this theorem in [3]:

Theorem 3. *Fix a 2-coloring of $\mathbb{E}^2$ and let T be a triangle with sides a, b, and c. Then T occurs monochromatically if and only if some equilateral triangle with side a, b, or c occurs monochromatically.*

Conjecture 1 is therefore equivalent to:

Conjecture 2. *Fix a 2-coloring of $\mathbb{E}^2$ and let T and T' be equilateral triangles with side lengths $d \neq d'$, respectively. Then at least one of T, T' occurs monochromatically.*

This is much stronger than any of the results above; in each of those conditional results, a list of three or more similar triangles is given, one of which must occur monochromatically. An intermediate problem would be to prove a conditional result like the ones above with a list of just two similar triangles; as far as we know, this has not been done even in the case of equilateral triangles.

2 Degenerate triangles in the plane

In Section 1 we discussed proper triangles in the plane; here we consider the case of degenerate triangles – that is, sets of three collinear points. In this section, an (a, b, c) triangle will refer to a triangle with side lengths a, b, and c (and as above, when we refer to a triangle in the plane, we really mean the set of its vertices).

For any collinear set S of 3 points, it is known that with 16 colors one can avoid a monochromatic copy of S in $\mathbb{E}^n$ for all n ([5]), but it is an open question if this is the best possible. Figure 4 shows that in the plane, it is possible to avoid the $(a, a, 2a)$ degenerate triangle with only 3 colors. This tiling extends to cover $\mathbb{E}^2$; each hexagon has diameter $2a$ and all of the hexagons are half-open as shown for the uppermost hexagon in Figure 4.

Proposition 1. *If χ is a 2-coloring of $\mathbb{E}^2$ that contains a monochromatic copy of the (a, a, a) triangle, then for any $b > 0$, χ also contains a monochromatic copy of the degenerate $(a, b, a + b)$ triangle.*

Proof. Let χ be a 2-coloring of $\mathbb{E}^2$ in the colors black and white, and suppose the three vertices of an (a, a, a) triangle in the plane are monochromatic, as in Figure 5 (all acute angles are $\pi/3$). Suppose by way of contradiction that we can avoid a monochromatic $(a, b, a + b)$ triangle. In the diagram, vertices A and B must then be colored white, forcing vertex C to be colored black. Then vertex E must be colored white. Since both E and B are white, it is impossible to color vertex D either black or white without producing a monochromatic $(a, b, a + b)$ triangle, thus completing the proof. $\square$

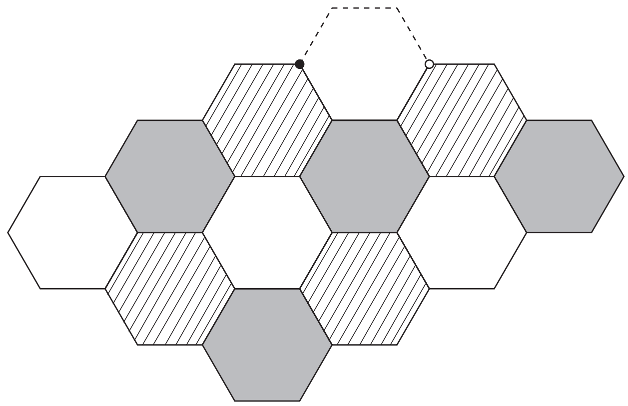

Figure 4: A sketch of the 3-coloring avoiding the $(a, a, 2a)$ triangle.

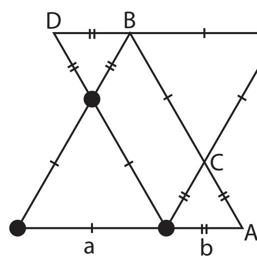

Figure 5: Sketch of the proof of Proposition 1.

Proposition 2. *If χ is a 2-coloring of $\mathbb{E}^2$, and for some $a, b > 0$, χ contains a monochromatic copy of the $(a+b, a+b, a+b)$ triangle, χ also contains a monochromatic copy of the degenerate $(a, b, a+b)$ triangle.*

Proof. Fix $a, b > 0$, and let χ be a 2-coloring of $\mathbb{E}^2$ in the colors black and white such that there is a monochromatic $(a+b, a+b, a+b)$ triangle, as in Figure 6 (again, all acute angles are $\pi/3$). Suppose by way of contradiction that we can avoid a monochromatic $(a, b, a+b)$ triangle. In the diagram, vertices A, B and C must be colored white, forcing vertex E to be colored black. Now, as in Proposition 1, it is impossible to color vertex D without producing a monochromatic $(a, b, a+b)$ triangle.

□

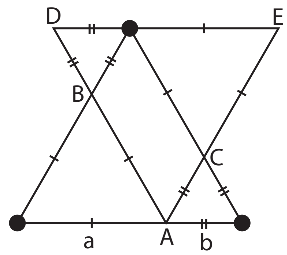

Figure 6: Sketch of the proof of Proposition 2.

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[5] E. Straus, A combinatorial theorem in group theory, *Math. Comp.* **29** (1975), 303–309.