Intersecting domino tilings

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1 Introduction

A typical problem in extremal set theory is to give conditions that a family of sets must satisfy, and then ask what is the maximal size of a family of sets which can be formed satisfying these conditions. One simple example is to insist that every two pairs of sets in your family intersect at least ℓ times, or in other words the family of sets is ℓ-intersecting. One of the most celebrated results in extremal set theory looks at the maximal size of an ℓ-intersecting family.

Theorem 1 (Erdős-Ko-Rado [5]). *Let $\mathcal{F}$ be an ℓ-intersecting family of sets, with each element A_i a k-element subset of $\{1,\ldots,n\}$. Then for $n \geq (k-\ell+1)(\ell+1)$*

$$|\mathcal{F}| \leq \binom{n-\ell}{k-\ell}.$$

In the original statement of the proof this was shown to hold for $n \geq n_0(k,\ell)$. Frankl [7] established the above bound for $\ell \geq 15$ and then Wilson [13] established the bound in general. Taking all k element sets containing $\{1,\ldots,\ell\}$ forms an ℓ-intersecting family of size $\binom{n-k}{k-\ell}$. Theorem 1 then says that this is essentially best possible, in other words you cannot be more clever than doing the obvious thing.

This result has been generalized to other combinatorial objects which share a notion of intersection. The type of objects that have previously been studied include permutations [4], set partitions [11], colored sets [2], arithmetic progressions [6], strings [8], and vector spaces [9]. In this note we will consider a new type of intersection problem, namely the intersection of tilings.

A tiling consists of covering a board using tile pieces from a given set so that the board is completely covered and no two tiles overlap (for more about tilings we recommend the excellent survey paper by Ardila and Stanley [1]). We say that two

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tilings of the board intersect if there is a tile placed in the same position on both boards. For example, Figure 1 shows two tilings of a 4×5 board using dominoes. The shaded tile is placed the same way in both tilings so these intersect.

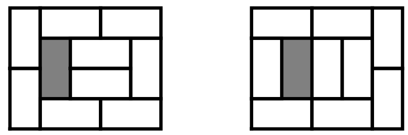

Figure 1: An example of intersecting tilings.

In this note we will find the maximal size of families of intersecting tilings for the cases of tiling the $2\times n$ strip (Section 2) and the $3\times(2n)$ strip (Section 3) by using dominoes.

2 Tilings of $2\times n$ using dominoes

It is well known that the number of tilings of the $2\times n$ strip using dominoes is $F(n+1)$ where $F(n)$ are the well known Fibonacci numbers, $F(1) = F(2) = 1$ and $F(n) = F(n-1) + F(n-2)$ (this is sequence `A000045` in the `OEIS` [12]).

Theorem 2. *Let $\mathcal{T}$ be an intersecting family of tilings of the $2\times n$ strip using dominoes. Then $|\mathcal{T}| \le F(n)$.*

Proof. We first note that by taking all the tilings of the $2\times n$ strip that begin with a vertical domino we have an intersecting family of size $F(n)$. So it remains to show that this cannot be improved upon.

Consider the graph which is formed by taking all possible tilings and putting an edge between two tilings it they do **not** intersect. The problem of finding a maximal intersecting family is equivalent to finding a maximal independent set in this graph. We can split the vertices into two sets $\mathcal{H}$ and $\mathcal{V}$. Where $\mathcal{H}$ is the $F(n-1)$ tilings that start with two horizontal tiles and $\mathcal{V}$ is the $F(n)$ tilings that start with a vertical tile. By definition, all edges in the graph are between $\mathcal{H}$ and $\mathcal{V}$ (i.e., the graph is bipartite).

We claim that there is a matching between $\mathcal{H}$ and a subset of $\mathcal{V}$. To see this, suppose that we have a tiling T in $\mathcal{H}$. Then we can decompose this tiling into a sequence of blocks where a block consists of two horizontal tiles followed by any number of vertical tiles. We now map $T \to S$ block by block using the rule shown in Figure 2. For any $T \in \mathcal{H}$ the resulting S will start with a vertical tile and so is in $\mathcal{V}$, further block by block it can be seen that S and T have no common tile, so there is

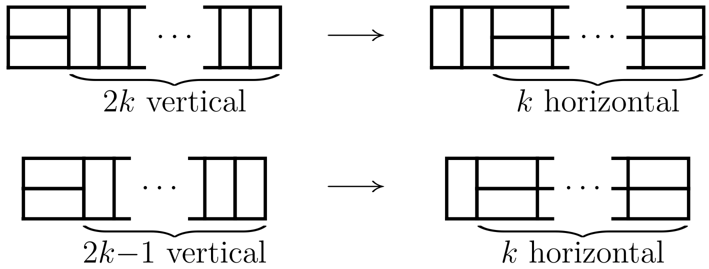

Figure 2: The rule for forming the matching between $\mathcal{H}$ and $\mathcal{V}$.

an edge between S and T. Finally it is easy to check that this map is 1-to-1, so gives our desired matching.

Since there is a matching from every element of $\mathcal{H}$ to an element of $\mathcal{V}$ it follows that for any subset $\mathcal{Q}$ of $\mathcal{H}$ that the number of elements in $\mathcal{V}$ adjacent to $\mathcal{Q}$ has size at least $|\mathcal{Q}|$. (This is the rarely used direction of Hall's Marriage Theorem.) Now suppose that $\mathcal{T}$ is an intersecting family and let $\mathcal{Q} = \mathcal{T} \cap \mathcal{H}$ and $\mathcal{R} = \mathcal{T} \cap \mathcal{V}$. Since the elements of $\mathcal{R}$ cannot be adjacent to elements of $\mathcal{Q}$ the above comment implies that $|\mathcal{R}| \le |\mathcal{V}| - |\mathcal{Q}|$. So we have

$$|\mathcal{T}| = |\mathcal{Q}| + |\mathcal{R}| \le |\mathcal{Q}| + (|\mathcal{V}| - |\mathcal{Q}|) = |\mathcal{V}| = F(n). \qquad \square$$

3 Tilings of $3\times(2n)$ using dominoes

We now turn to tilings of the $3\times(2n)$ board. We first count the number of such tilings (this has been done previously and is `A001835` in the `OEIS` [12]). A commonly used approach is to set up a system of linear recurrences and then solve the system, we will do a variation where we count the number of weighted walks in a small graph.

The basic idea is to break the $3\times(2n)$ strip into n small blocks of size 3×2, and consider how horizontal dominoes can intersect the break between consecutive blocks. Since the area of each block is even, it follows that in the breaks we must have an even number of horizontal dominoes. This gives the four possibilities shown in Figure 3, the fourth of which cannot happen in a tiling of $3\times(2n)$, we will refer to the remaining possibilities, from left to right, as $\shortmid$, $\pm$ and $\mp$.

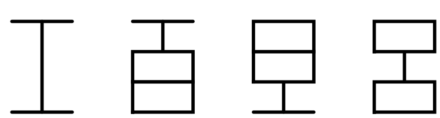

Figure 3: The different configuration of horizontal dominoes between blocks.

To count the total number of tilings we can take all possible configurations of horizontal dominoes in the breaks and then count the ways to fill in the remaining untiled portion of the strip. We can do this by using weighted walks in a small directed graph where the vertex set is $\shortmid$, $\underline{\perp}$, $\overline{\top}$ and the weight of an edge is the number of ways to fill in the unused area of a block between the two column breaks indicated. For instance there are 3 ways to fill in a 3×2 strip so there is a loop of weight 3 for the edge $\shortmid\,\shortmid$. Similarly, edges $\underline{\perp}\underline{\perp}$, $\overline{\top}\overline{\top}$, $\shortmid\underline{\perp}$, $\underline{\perp}\shortmid$, $\shortmid\overline{\top}$ and $\overline{\top}\shortmid$ have weight 1 since there is only one way to fill in the block, while $\underline{\perp}\overline{\top}$ and $\overline{\top}\underline{\perp}$ have weight 0 since there is no way to fill in the uncovered area using dominoes. This gives us the following adjacency matrix for the graph.

$$A = \begin{array}{c} \\ \shortmid \\ \underline{\perp} \\ \overline{\top} \end{array}\!\!\begin{array}{c} \begin{array}{ccc} \shortmid & \underline{\perp} & \overline{\top} \end{array} \\ \begin{pmatrix} 3 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \end{array}$$

Since the left and right sides of the $3\times(2n)$ board correspond to $\shortmid$ we need to find the sum of the weight of walks of length n in the graph that start and end at $\shortmid$. This is equivalent to finding the $(1,1)$ entry of A^n. The eigenvalues of A are $2+\sqrt{3}$, $2-\sqrt{3}$ and 1, using these along with their eigenvectors to form projection matrices we have

$$A^n = \big(2+\sqrt{3}\,\big)^n \begin{pmatrix} \frac{3+\sqrt{3}}{6} & \frac{\sqrt{3}}{6} & \frac{\sqrt{3}}{6} \\ \frac{\sqrt{3}}{6} & \frac{3-\sqrt{3}}{12} & \frac{3-\sqrt{3}}{12} \\ \frac{\sqrt{3}}{6} & \frac{3-\sqrt{3}}{12} & \frac{3-\sqrt{3}}{12} \end{pmatrix} + \big(2-\sqrt{3}\,\big)^n \begin{pmatrix} \frac{3-\sqrt{3}}{6} & -\frac{\sqrt{3}}{6} & -\frac{\sqrt{3}}{6} \\ -\frac{\sqrt{3}}{6} & \frac{3+\sqrt{3}}{12} & \frac{3+\sqrt{3}}{12} \\ -\frac{\sqrt{3}}{6} & \frac{3+\sqrt{3}}{12} & \frac{3+\sqrt{3}}{12} \end{pmatrix} + 1^n \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2} \\ 0 & -\frac{1}{2} & \frac{1}{2} \end{pmatrix}.$$

Taking the sum of the $(1,1)$ entries we have established the following.

Proposition 3. *It T_n is the number of tilings of $3\times(2n)$ by dominoes then*

$$T_n = \frac{3+\sqrt{3}}{6}\big(2+\sqrt{3}\,\big)^n + \frac{3-\sqrt{3}}{6}\big(2-\sqrt{3}\,\big)^n.$$

Looking at the possible forms of the 3×2 blocks we get nine possible shapes (note that nine is also the sum of the entries of A). These are shown in Figure 4. The tiles split into three groups, "blue" tiles with a single horizontal domino on the top, "red" tiles with a single horizontal domino on the bottom and a universal tile. Since *every* 3×2 block has at least one horizontal domino then any $3\times(2n)$ tiling which uses a

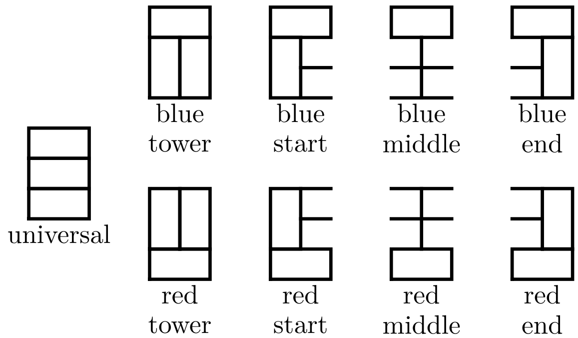

Figure 4: The possible 3×2 blocks.

universal tile will intersect every other tiling, i.e., it will be universally intersecting. It turns out that these are the only universally intersecting configurations.

We now count the number of tilings that do not have a universal tile. The previous approach is easily adopted and the only change is to remove a single possibility between $\vert\ \vert$, namely the one with three horizontal dominoes. This gives the following matrix

$$B = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}.$$

This matrix has eigenvalues 3, 1 and 0, so that for $n \geq 1$ we have, similarly to before,

$$B^n = 3^n \begin{pmatrix} \frac{2}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{6} & \frac{1}{6} \\ \frac{1}{3} & \frac{1}{6} & \frac{1}{6} \end{pmatrix} + 1^n \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2} \\ 0 & -\frac{1}{2} & \frac{1}{2} \end{pmatrix}.$$

Taking the sum of the $(1,1)$ entries we have the following.

Proposition 4. *If S_n is the number of tilings of $3{\times}(2n)$ by dominoes which does not have three horizontal dominoes in a column, then $S_n = 2{\cdot}3^{n-1}$.*

We are now ready to bound the size of a maximal intersecting family.

Theorem 5. *Let $\mathcal{T}$ be an intersecting family of tilings of the $3{\times}(2n)$ strip using dominoes. Then*

$$|\mathcal{T}| \leq \frac{3+\sqrt{3}}{6}\big(2+\sqrt{3}\,\big)^n + \frac{3-\sqrt{3}}{6}\big(2-\sqrt{3}\,\big)^n - 3^{n-1}.$$

Proof. As in the $2{\times}n$ case we form a graph where each vertex is a tile and two vertices are connected if they do not intersect. Any tiling which contains the universal tile

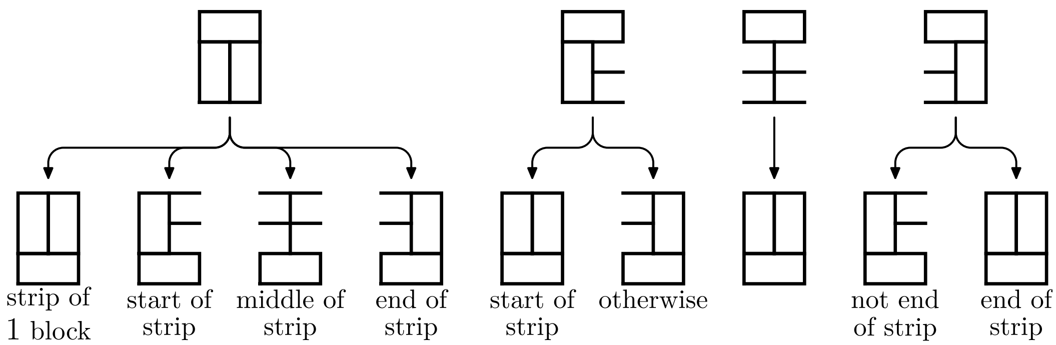

Figure 5: Rule for mapping in the $3\times(2n)$ case.

will be an isolated vertex. The remaining tiles can be split into two groups, those that start with a red tile and those that start with a blue tile. As before this makes the graph into a bipartite graph.

Claim. *There is a perfect matching in the set of tilings which do not contain the universal tile.*

Before we prove the claim let us show how this will give the statement of the theorem. In an intersecting family we can take any number of the isolated vertices and at most one of the tilings in each edge of the perfect matching. There are $T_n - S_n$ isolated vertices and $\frac{1}{2}S_n$ edges in the perfect matching. So that an intersecting family has at most $T_n - \frac{1}{2}S_n$. Now using the results from Propositions 3 and 4 the result will follow.

To prove the claim we give a mapping between tilings that start with a blue tile to tilings that start with a red tile. So let T be a tiling. We break T into (maximal) blue and red strips. It suffices to give a mapping that takes a blue strip into a red strip of the same size (and vice-versa) which does not intersect. Such a mapping is given in Figure 5. It is easy to check that this mapping satisfies all the needed properties and concludes the proof of the theorem. $\square$

4 Concluding remarks

Tiling problems have been very popular (both in looking at existence and enumeration of tilings). Looking for maximal intersecting family of tilings opens up an entirely new avenue of investigation of tilings. In this note we have restricted ourselves to domino tilings of the $2\times n$ and $3\times 2n$ boards but one can more generally look at domino tilings of $k\times n$ boards.

Besides looking at domino tilings one can consider tilings with squares and dominoes, or squares and “L”s [3], or tetris pieces, or polyominoes (see Golomb's [10] excellent book on the subject which also deals extensively with tiling problems), or hexagonal animals, or three-dimensional tilings. For each problem one can also consider a variety of different board configurations. The possibilities of different problems are limited only by the imagination.

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